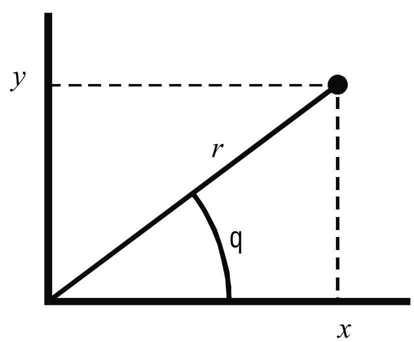
Circular membrane

When we studied the one-dimensional wave equation we found that the method of separation of variables resulted in two simple harmonic oscillator (ordinary) differential equations. The solutions of these were relatively straightforward. Here we are interested in the next level of complexity – when the ODEs which arise upon separation may be different from the familiar SHO equation. This complexity arises when non-cartesian coordinate systems are used.

Symmetry and coordinate systems
The appropriate coordinate system is chosen so as to reflect the symmetry of the system. Using the appropriate coordinate system often results in a simplification of the equations to be solved. And a further benefit appears at the stage where the boundary conditions are invoked. Thus, for example, the flow of heat in a cylindrical pipe is best treated by expressing the diffusion equation in cylindrical polar coordinates; the vibrations of a sphere are best treated by writing the wave equation in spherical polar coordinates; the vibration of a circular drum head is best treated in terms of the wave equation written in plane polar coordinates.

Note that in all these cases it is the Laplacian operator ∇^2 which must be expressed in the chosen coordinate system; you would look this up in a good reference book.

Boundary and initial conditions
Use plane polar coordinates – coordinate system determined by the boundary conditions.



$$x = r\cos\theta$$
$$y = r\sin\theta$$

plane polar coordinates

The motion of the membrane is described by the wave equation (in two spatial dimensions):

$$\nabla^2\Psi - \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} = 0\ .$$

We know the form of ∇^2 in rectangular cartesian coordinates; here we need to transform it to polar coordinates. This gives

$$\begin{aligned}\nabla^2\Psi &= \frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}\\ &= \frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r}\frac{\partial\Psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}\\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2},\end{aligned}$$

which may be obtained by application of the chain rule for differentiation. The second and third lines are equivalent; you should use whichever is more convenient.

(*You should be assured that a professional physicist would not consider deriving the expression for* ∇^2 *in different coordinate systems; he/she would look this sort of thing up in a good reference book.*)

In plane polar coordinates the wave equation becomes

$$\frac{\partial^2\Psi(r,\theta,t)}{\partial r^2}+\frac{1}{r}\frac{\partial\Psi(r,\theta,t)}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Psi(r,\theta,t)}{\partial\theta^2}-\frac{1}{v^2}\frac{\partial^2\Psi(r,\theta,t)}{\partial t^2}=0\,.$$

The field variable Ψ is a function of the two spatial variables, here r and θ, together with the time variable t. That is,

$$\Psi=\Psi(r,\theta,t)\,.$$

But for the purposes of this exercise we shall be concerned with the solutions which exhibit *circular symmetry*. That is, we are considering solutions that have the symmetry of the boundary conditions.

(*We note that in general the solutions of a PDE have symmetry lower than or equal to that of the boundary conditions.*)

The circularly symmetric solutions to the wave equation do not depend on θ; they depend only on r and t. In other words a symmetry (in the solution) is indicated by the vanishing of a coodinate from the field variable. So here we are concerned with the field variable $\Psi(r,t)$. And since this Ψ does not depend on θ the wave equation reduces to

$$\frac{\partial^2\Psi(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial\Psi(r,t)}{\partial r}-\frac{1}{v^2}\frac{\partial^2\Psi(r,t)}{\partial t^2}=0\,.$$

This is a partial differential equation in *two* independent variables. Thus we expect to be able to separate this into two ordinary differential equations.

Separation of variables

The procedure for separation of variables uses the substitution

$$\Psi(r,t)=R(r)T(t)\,.$$

When this is inserted into the wave equation we obtain

$$\frac{\mathrm{d}^2R}{\mathrm{d}r^2}T+\frac{1}{r}T\frac{\mathrm{d}R}{\mathrm{d}r}-\frac{1}{v^2}R\frac{\mathrm{d}^2T}{\mathrm{d}t^2}=0$$

and we see that the differentials are now total differentials. The next step is to divide through by $\Psi(r,t)=R(r)T(t)$. This gives:

$$\frac{1}{R}\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+\frac{1}{rR}\frac{\mathrm{d}R}{\mathrm{d}r}=\frac{1}{v^2T}\frac{\mathrm{d}^2T}{\mathrm{d}t^2}.$$

Here the left and right hand sides are independent and this can only be so if they are equal to a constant. For convenience we will write this constant as $-k^2$; we shall see that this will help in satisfying the boundary conditions (recall the choice and justification of a similar separation constant for the vibrating string). Using this separation constant gives us two ordinary differential equations

$$\frac{\mathrm{d}^2T}{\mathrm{d}t^2}+k^2v^2T=0$$

and

$$\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}r}+k^2R=0.$$

The first equation, for the time function, is nothing new; it is the simple harmonic oscillator equation. This has solutions

$$T_k\left(t\right)=A\cos\left(kvt\right)+B\sin\left(kvt\right),$$

although at this stage we know nothing about the allowed values of the separation constants *k*. We shall see how the allowed *k* values follow from the boundary conditions.

The second equation, for the radial function, is new. It is a linear ordinary differential equation but it does *not* have constant coefficients. This makes the solutions a little more complicated. We can make a simplification through the substitution $s=kr$. This will remove the variable *k* from the equation. The substitution gives us an equation for *R* as a function of *s*:

$$\frac{\mathrm{d}^2R}{\mathrm{d}s^2}+\frac{1}{s}\frac{\mathrm{d}R}{\mathrm{d}s}+R=0.$$

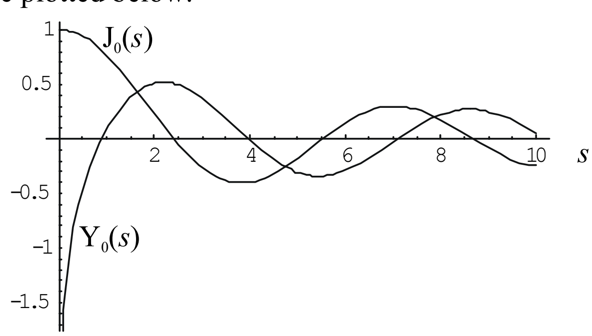
This is a special case of Bessel's equation; we shall investigate *how* to solve this equation later in the course. For the present we note that Mathematica *is* able to solve the equation and it gives the solution as

R = *A* BesselJ[0, *s*] + *B* BesselY[0, *s*].

Here *A* and *B* are constants and they multiply Bessel functions, which Mathematica represents by BesselJ[0, *s*] and BesselY[0, *s*]. The conventional notation for these Bessel functions is

$$\mathrm{J}_0\left(s\right)\ \text{ and }\ \mathrm{Y}_0\left(s\right).$$

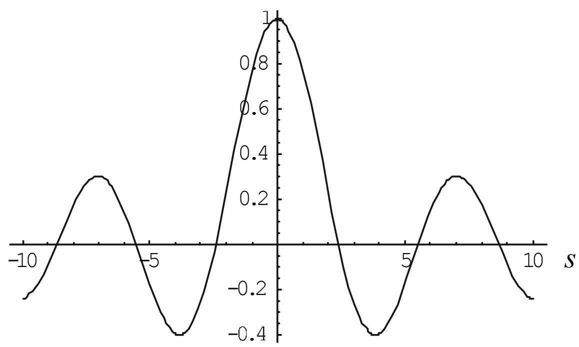
These functions are plotted below.



Bessel functions $\mathrm{J}_0\left(s\right)$ and $\mathrm{Y}_0\left(s\right)$

You should note that the value of k determines the *horizontal scaling* of the functions.

There *have* to be two independent solutions to the Bessel equation since it is of second order. However we see that in this problem – the circular drum head – the Y_0 functions are not allowed as they go to minus infinity as $r \to 0$ which is unphysical; only the J_0 functions are permitted as they are finite at $r = 0$. The profile of the disturbance across the drumhead will then be described by functions of the form



profile of disturbance across a drum head

Boundary conditions
The boundary condition is that the edge of the drum head is fixed so that it cannot be displaced. This means that

$$J_0(kR) = 0$$

where R is the radius of the drum (not the $R(r)$ function here).

This determines the allowed values of k since kR must correspond to a zero of the Bessel function. In other words, if α_m are the values of s for which $J_0(s)$ is zero, then it follows that the allowed values of k are

$$k_m = \alpha_m / R.$$

(Note how the *boundary conditions* lead to *quantisation.*)

Complete solution
We have now found that there is a whole set of radial functions $R(r)$ that are possible solutions of our problem:

$$R_m(r) = J_0\left(\frac{\alpha_m}{R} r\right).$$

Then combining this with the solution to the time equation, we have the m^{th} solution as

$$\Psi_m(r,t) = \left\{a_m \cos(k_m vt) + b_m \sin(k_m vt)\right\} J_0\left(\frac{\alpha_m}{R} r\right)$$

and the general solution is a linear combination of all these

$$\Psi(r,t) = \sum_{m=1}^{\infty} \left\{a_m \cos(k_m vt) + b_m \sin(k_m vt)\right\} J_0\left(\frac{\alpha_m}{R} r\right).$$

Initial conditions

The coefficients a_m and b_m are determined form the initial conditions. The initial conditions are specified as

$$\Psi(r,0) = f(r)$$

$$\left.\frac{\partial\Psi}{\partial t}\right|_{t=0} = g(r).$$

We substitute the general solution above into these expressions to give

$$\sum_{m=1}^{\infty} a_m \mathrm{J}_0\left(\frac{\alpha_m}{R} r\right) = f(r)$$

$$\sum_{m=1}^{\infty} b_m k_m v\, \mathrm{J}_0\left(\frac{\alpha_m}{R} r\right) = g(r).$$

These expressions give the initial functions $f(r)$ and $g(r)$ in terms of the coefficients a_m and b_m. What we want to do is to *invert* this result to give the coefficients a_m and b_m in terms of the initial functions $f(r)$ and $g(r)$. The key to doing this is *orthogonality*. – as we shall see in future lectures. And when these coefficients are found in this way then the complete solution for the (circularly symmetric) behaviour of the drum head is given by

$$\Psi(r,t) = \sum_{m=1}^{\infty} \left\{a_m \cos(k_m vt) + b_m \sin(k_m vt)\right\} \mathrm{J}_0\left(\frac{\alpha_m}{R} r\right).$$